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A study of new hypergeometric transformations

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Abstract
The aim of this research paper is to establish a quite general transformation involving hypergeometric functions by the method of elementary manipulation of the series representations. Certain known, as well as new hypergeometric transformations and identities not previously recorded in the literature, are then deduced by means of the generalized Kummer and Dixon theorems obtained earlier by Lavoie *et al*. An application of a newly obtained identity is also given.

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1. Introduction and preliminaries

The generalized hypergeometric function given by

$$ {}_AF_B\left[\begin{matrix} a_1, & \ldots, & a_A \\ & & & ; & x \\ b_1, & \ldots, & b_B \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_A)_n}{(b_1)_n \cdots (b_B)_n} \frac{x^n}{n!} \tag{1.1} $$

has been discussed at great length by numerous authors such as Slater [14] and Exton [2], who have treated many of its properties, including convergence of its series representation (1.1). Here

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. $$

Many hypergeometric identities are known (see Slater [14]) and Prudnikov *et al* [12]). Most of them are the summation formulae for hypergeometric series, either infinite series or

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terminating series including binomial coefficient summations. Recent work has concentrated on developing techniques for verifying asserted or conjectured identities, rather than deriving new ones.

It is well known that the classical summation theorems, such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; and those of Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$, play an important role in the theory of generalized hypergeometric series. It should be remarked here that whenever hypergeometric functions reduce to gamma functions, the results are very useful and have various applications.

The above-mentioned classical summation theorems are listed below, so that the paper may be self-contained.

Gauss theorem

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ;\ 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

for $\operatorname{Re}(c-a-b) > 0$.

Kummer theorem

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & 1+a-b \end{matrix} ;\ -1\right] = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{1}{2}a\right)}{\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma(1+a)}. \tag{1.2} $$

Gauss second theorem

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & \frac{1}{2}(a+b+1) \end{matrix} ;\ \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}. \tag{1.3} $$

Bailey theorem

$$ {}_2F_1\left[\begin{matrix} a, & 1-a \\ & c \end{matrix} ;\ \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}. \tag{1.4} $$

Watson theorem

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c & \end{matrix} ;\ 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \tag{1.5} $$

for $\operatorname{Re}(2c-a-b) > -1$.

Dixon theorem

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c & \end{matrix} ;\ 1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)} \tag{1.6} $$

for $\operatorname{Re}(a-2b-2c) > -2$.

Whipple theorem

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & e, & f \end{matrix} ; 1\right] = \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma\left(\frac{1}{2}a+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}f\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}f\right)} \tag{1.7}
$$

with $e+f=2c+1$, $a+b=1$ and $\mathrm{Re}(e+f-a-b-c)>0$.

Saalschütz theorem

$$
{}_3F_2\left[\begin{matrix} -n, & a, & b \\ & c, & 1+a+b-c-n \end{matrix} ; 1\right] = \frac{(c)_n(c-a-b)_n}{(c-a)_n(c-b)_n}.
$$

Recently good progress has been done in generalizing a few of the above-mentioned classical summation theorems. In fact, in a series of three papers, Lavoie *et al* [5–7] have obtained explicit expressions of

$$
{}_2F_1\left[\begin{matrix} a, & b \\ & 1+a-b+i \end{matrix} ; -1\right], \tag{1.8}
$$

$$
{}_2F_1\left[\begin{matrix} a, & b \\ & \frac{1}{2}(1+a+b+i) \end{matrix} ; \frac{1}{2}\right] \tag{1.9}
$$

and

$$
{}_2F_1\left[\begin{matrix} a, & 1-a+i \\ & c \end{matrix} ; \frac{1}{2}\right] \tag{1.10}
$$

for each $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, as well as

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & \frac{1}{2}(a+b+i+1), & 2c+j \end{matrix} ; 1\right], \tag{1.11}
$$

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+a-b+i, & 1+a-c+i+j \end{matrix} ; 1\right] \tag{1.12}
$$

and

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & e, & f \end{matrix} ; 1\right], \tag{1.13}
$$

with $a+b=1+i$ and $e+f=2c+i+j$ for each $i, j = 0, \pm 1, \pm 2, \pm 3$.

For $i = 0$, equations (1.8)–(1.10) reduce to Kummer (1.2), Gauss second (1.3) and Bailey (1.4) summation theorems, respectively. For $i = j = 0$, equations (1.11)–(1.13) reduce to Watson (1.5), Dixon (1.6) and Whipple (1.7) summation theorems, respectively. It is not out

Table 1. Coefficients of A_i and B_i.

i	A_i	B_i
0	1	0
1	-1	1
2	$1+a-b$	-2
3	$3b-2a-5$	$2a-b+1$
4	$2(a-b+3)(1+a-b)-(b-1)(b-4)$	$-4(a-b+2)$
5	$-4(6+a-b)^2+2b(6+a-b)+b^2$ $+22(6+a-b)-13b-20$	$4(6+a-b)^2+2b(6+a-b)-b^2$ $-34(6+a-b)-b+62$

of place to mention here that in 1997, Lewanowicz [9] gave the extensions of (1.11)–(1.13) in the most general case for arbitrary i and fixed $j\,(i, j \in \mathbb{Z})$.

In [10], Maier gave a relation between the $r = 2$ case of the Slater formula for ${}_{r+1}F_r(1)$ and a 3-parameter summation with linear parametric restrictions. He also derived two 3-parameter ${}_3F_2(1)$ summations with nonlinear parametric restrictions.

Guo *et al* [4], showed that several terminating summation and transformation formulae for basic hypergeometric series can be proved in a straightforward way. They also proved new finite forms of Jacobi triple product identity and Watson quintuple product identity. In [8], Lin *et al* derived a family of generating relations for a general polynomial system by applying a quadratic transformation to the Gauss hypergeometric series. Vidunas [15] classified the algebraic transformation of hypergeometric functions by finding all pull-back transformations of their hypergeometric equations to other hypergeometric equation.

The aim of this research paper is to establish a quite general transformation involving hypergeometric functions by the method of elementary manipulation of the series. Certain known, as well as identities not previously recorded in the literature, are then deduced by means of the generalized Kummer and Dixon theorems obtained by Lavoie *et al* [6, 7]. The results derived in the paper are simple, easily established and may be useful.

The following special cases of the generalizations of the classical Kummer summation theorem (which can be obtained from equation (6) given in [7] and Euler transformation [13, theorem 20, p 60]) and of the Dixon summation theorem obtained earlier by Lavoie *et al* [6, 7] will be required in our present investigation.

Generalized Kummer's theorem

$$
{}_2F_1\left[\begin{matrix} a, & b \\ & 1+a-b+i \end{matrix} ; -1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-b)\Gamma(1+a-b+i)}{2^a\Gamma(1-b+i))}
\times\left\{\frac{A_i}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+1\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-\left\lfloor\frac{1+i}{2}\right\rfloor\right)}
+\frac{B_i}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left\lfloor\frac{i}{2}\right\rfloor\right)}\right\} \qquad (1.14)
$$

for $i = 0, 1, 2, 3, 4, 5$. The coefficients A_i and B_i are given in table 1.

Generalized Dixon's theorem

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+a-b, & 1+a-c+i \end{matrix} ; 1\right]
$$

Table 2. Coefficients of A_i and B_i.

i	A_i	B_i
0	1	0
1	-1	1
2	$\frac{1}{2}\{(a-b-c+1)^2+(c-1)(c-3)-b^2+a\}$	-2
3	$3ab+c(a-b-c+4)$ $-(a+1)(a+2)-(a-1)(b-1)$	$(a+2)(a+4)-b(2a+5)$ $-3c(a-b-c+4)+3$

$$= \frac{2^{-2c+i}\Gamma(1+a-b)\Gamma(1+a-c+i)\Gamma(c-i)}{\Gamma(a-2c+i+1)\Gamma(a-b-c+i+1)\Gamma(c)}$$
$$\times \left\{A_i \frac{\Gamma\left(\frac{1}{2}a-c+\frac{1}{2}+\left\lfloor\frac{i+1}{2}\right\rfloor\right)\Gamma\left(\frac{1}{2}a-b-c+1+\left\lfloor\frac{i+1}{2}\right\rfloor\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+1\right)}\right.$$
$$\left.+B_i \frac{\Gamma\left(\frac{1}{2}a-c+1+\left\lfloor\frac{i}{2}\right\rfloor\right)\Gamma\left(\frac{1}{2}a-b-c+\frac{3}{2}+\left\lfloor\frac{i}{2}\right\rfloor\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}\right)}\right\} \qquad (1.15)$$

for $i = 0, 1, 2, 3$.

In both the results (1.14) and (1.15), $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients A_i and B_i are given in table 2.

The remainder of the paper is organized as follows. The main transformation formula and its derivation appear in section 2. Special cases of this transformation formula are included in section 3. Some of these special cases are believed to be new. Application of one of the new transformations presented in this paper appears in section 4.

2. Main transformation formula

The general transformation to be established is

$$ {}_{G+2}F_{H+2}\left[\begin{matrix} b, & c, & g_1, & \ldots, & g_G \\ f-b+i, & f-c+i+j, & h_1, & \ldots, & h_H \end{matrix} ; \; y\right]$$
$$= \frac{\Gamma(f-b+i)\Gamma(f-c+i+j)}{\Gamma(f+i)\Gamma(f-b-c+i+j)} \sum_{n=0}^{\infty} \frac{(b)_n(c-j)_n}{(f+i)_n n!}$$
$${}_{G+2}F_{H+2}\left[\begin{matrix} -n, & c, & g_1, & \ldots, & g_G \\ c-j, & f+i+n, & h_1, & \ldots, & h_H \end{matrix} ; \; -y\right] \qquad (2.1)$$

for $i, j = 0, 1, 2, \ldots$.

2.1. Derivation

In order to derive (2.1), we proceed as follows. Starting with the left-hand side of (2.1), denoting it by S and expressing ${}_{G+2}F_{H+2}$ as a series, we have

$$S = \sum_{m=0}^{\infty} \frac{(b)_m(c)_m(g_1)_m\cdots(g_G)_m}{(f-b+i)_m(f-c+i+j)_m(h_1)_m\cdots(h_H)_m}\frac{y^m}{m!},$$

which can be written as

$$S = \frac{\Gamma(f-b+i)\Gamma(f-c+i+j)}{\Gamma(f+i)\Gamma(f-b-c+i+j)} \times \sum_{m=0}^{\infty} \frac{(b)_m (c)_m (g_1)_m \cdots (g_G)_m}{(f+i)_{2m} (h_1)_m \cdots (h_H)_m m!} y^m$$
$$\cdot \frac{\Gamma(f+i+2m)\Gamma(f-b-c+i+j)}{\Gamma(f-b+i+m)\Gamma(f-c+i+j+m)}.$$

Using the Gauss theorem we will have

$$S = \frac{\Gamma(f-b+i)\Gamma(f-c+i+j)}{\Gamma(f+i)\Gamma(f-b-c+i+j)} \times \sum_{m=0}^{\infty} \frac{(b)_m (c)_m (g_1)_m \cdots (g_G)_m}{(f+i)_{2m} (h_1)_m \cdots (h_H)_m m!} y^m$$
$$\cdot {}_2F_1 \left[\begin{matrix} b+m, & c+m-j \\ & f+i+2m \end{matrix} ; 1 \right].$$

Now, expressing ${}_2F_1$ as a series and using $(b)_m (b+m)_n = (b)_{m+n}$, we will have

$$S = \frac{\Gamma(f-b+i)\Gamma(f-c+i+j)}{\Gamma(f+i)\Gamma(f-b-c+i+j)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(g_1)_m \cdots (g_G)_m (b)_{m+n} (c)_m (c-j)_{m+n}}{(h_1)_m \cdots (h_H)_m (c-j)_m (f+i)_{2m+n}} \frac{y^m}{m! n!}.$$

Changing n to $n-m$ and using the results of [13, equation (1), p 56];

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k)$$

and

$$(n-m)! = \frac{(-1)^m n!}{(-n)_m}, \qquad 0 \leqslant m \leqslant n;$$

and simplification, we will have

$$S = \frac{\Gamma(f-b+i)\Gamma(f-c+i+j)}{\Gamma(f+i)\Gamma(f-b-c+i+j)} \sum_{n=0}^{\infty} \frac{(b)_n (c-j)_n}{(f+i)_n n!} \sum_{m=0}^{n} \frac{(g_1)_m \cdots (g_G)_m}{(h_1)_m \cdots (h_H)_m}$$
$$\cdot \frac{(c)_m (-n)_m (-y)^m}{(f+i+n)_m (c-j)_m m!}.$$

Summing up the inner series, we finally have

$$S = \frac{\Gamma(f-b+i)\Gamma(f-c+i+j)}{\Gamma(f+i)\Gamma(f-b-c+i+j)} \sum_{n=0}^{\infty} \frac{(b)_n (c-j)_n}{(f+i)_n n!}$$
$$\times {}_{G+2}F_{H+2} \left[\begin{matrix} -n, & c, & g_1, & \ldots, & g_G \\ c-j, & f+i+n, & h_1, & \ldots, & h_H \end{matrix} ; -y \right].$$

This completes the proof of (2.1).

Remarks

(1) The result (2.1) is a special case of a general result given by Slater [14, equation (2.4.10), p 60], and obtained by other means.

(2) In (2.1), if we take $j = 0$, we obtain a result due to Exton [3, equation (1.6), p 136].

3. New transformations and identities

In this section, we shall mention some of the very interesting special cases of our main transformation formula (2.1).

(a) In (2.1), if we take $G = 1$, $H = 0$, $g_1 = a$ and $f = 1 + a$, we have

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+a-b+i, & 1+a-c+i+j \end{matrix} ; \ y\right] $$

$$ = \frac{\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)}{\Gamma(1+a+i)\Gamma(1+a-b-c+i+j)} \sum_{n=0}^{\infty} \frac{(b)_n (c-j)_n}{(1+a+i)_n n!} $$

$$ {}_3F_2\left[\begin{matrix} -n, & a, & c \\ & c-j, & 1+a+i+n \end{matrix} ; \ -y\right]. \tag{3.1} $$

Taking $i = 0$, $j = 1$ and $y = 1$, we will have

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+a-b, & 2+a-c \end{matrix} ; \ 1\right] $$

$$ = \frac{\Gamma(1+a-b)\Gamma(2+a-c)}{\Gamma(1+a)\Gamma(2+a-b-c)} \sum_{n=0}^{\infty} \frac{(b)_n (c-1)_n}{(1+a)_n n!} $$

$$ \times {}_3F_2\left[\begin{matrix} -n, & a, & c \\ & 1+a+n, & c-1 \end{matrix} ; \ -1\right]. $$

Expressing ${}_3F_2$ appearing on the right-hand side as a series, noting that

$$ \frac{(c)_m}{(c-1)_m} = 1 + \frac{m}{(c-1)}, $$

then separating into two parts, summing up the series and using Kummer theorem, we have

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+a-b, & 2+a-c \end{matrix} ; \ 1\right] = \frac{\Gamma(1+a-b)\Gamma(2+a-c)}{\Gamma(1+a)\Gamma(2+a-b-c)} $$

$$ \times \sum_{n=0}^{\infty} \frac{(b)_n (c-1)_n}{(1+a)_n n!} \left[\left(1 - \frac{a}{2(c-1)}\right) \frac{\Gamma(1+a+n)\Gamma\left(1+\frac{1}{2}a\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a+n\right)} \right. $$

$$ \left. + \frac{a}{2(c-1)} \frac{\Gamma(1+a+n)\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)}{\Gamma(1+a)\Gamma\left(\frac{1}{2}+\frac{1}{2}a+n\right)} \right]. $$

Summing up the two series, using the Gauss summation theorem, and simplifying, we obtain a known result due to Lavoie *et al* [6, equation (2), $i = 0$, $j = 1$, p 268]. Similarly, other results given in [6] can be obtained from our result (3.1). On the other hand, taking $j = 0$ and $y = -1$ in (3.1), we get

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+a-b+i, & 1+a-c+i \end{matrix} ; \ -1\right] $$

$$= \frac{\Gamma(1+a-b+i)\Gamma(1+a-c+i)}{\Gamma(1+a+i)\Gamma(1+a-b-c+i)} \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(1+a+i)_n n!}$$

$$\times {}_2F_1 \left[\begin{matrix} -n, & a \\ & 1+a+n+i \end{matrix} ; \ 1 \right].$$

Using the Gauss summation theorem, simplifying and summation of the series, we obtain

$${}_3F_2 \left[\begin{matrix} a, & b, & c \\ & 1+a-b+i, & 1+a-c+i \end{matrix} ; \ -1 \right]$$

$$= \frac{\Gamma(1+a-b+i)\Gamma(1+a-c+i)}{\Gamma(1+a+i)\Gamma(1+a-b-c+i)}$$

$$\times {}_4F_3 \left[\begin{matrix} b, & c, & \frac{1}{2}+\frac{1}{2}i, & 1+\frac{1}{2}i \\ & 1+i, & \frac{1}{2}+\frac{1}{2}a+\frac{1}{2}i, & 1+\frac{1}{2}a+\frac{1}{2}i \end{matrix} ; \ 1 \right] \tag{3.2}$$

for $i = 0, 1, 2, \ldots$. This is a very general and new result not previously appeared in the literature.

In (3.2), if we put $i = 0$, we obtain a known result recorded in [12, equation (2), p 546].

In (3.2), if we put $i = 1$, we obtain

$${}_3F_2 \left[\begin{matrix} a, & b, & c \\ & 2+a-b, & 2+a-c \end{matrix} ; \ -1 \right] = \frac{\Gamma(2+a-b)\Gamma(2+a-c)}{\Gamma(2+a)\Gamma(2+a-b-c)}$$

$$\times {}_4F_3 \left[\begin{matrix} b, & c, & 1, & \frac{3}{2} \\ & 2, & 1+\frac{1}{2}a, & \frac{3}{2}+\frac{1}{2}a \end{matrix} ; \ 1 \right]. \tag{3.3}$$

An application of this new result will be given in section 4.

Further in (3.1), if we take $j = 0$ and $y = 1$, we get

$${}_3F_2 \left[\begin{matrix} a, & b, & c \\ & 1+a-b+i, & 1+a-c+i \end{matrix} ; \ 1 \right]$$

$$= \frac{\Gamma(1+a-b+i)\Gamma(1+a-c+i)}{\Gamma(1+a+i)\Gamma(1+a-b-c+i)} \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(1+a+i)_n n!}$$

$$\times {}_2F_1 \left[\begin{matrix} -n, & a \\ & 1+a+i+n \end{matrix} ; \ -1 \right] \tag{3.4}$$

for $i = 0, 1, 2, 3, 4, 5$.

Applying the generalized Kummer theorem (1.14) on (3.4), we will have

$${}_3F_2 \left[\begin{matrix} a, & b, & c \\ & 1+a-b+i, & 1+a-c+i \end{matrix} ; \ 1 \right]$$

$$= \frac{\Gamma(1+a-b+i)\Gamma(1+a-c+i)}{\Gamma(1+a+i)\Gamma(1+a-b-c+i)} \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(1+a+i)_n n!} \times \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1+n)\Gamma(1+a+i+n)}{2^a \Gamma(1+n+i)} \times \left\{ \frac{A_i}{\Gamma\left(\frac{1}{2}a+n+\frac{1}{2}i+1\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}+\left\lfloor\frac{i+1}{2}\right\rfloor\right)} + \frac{B_i}{\Gamma\left(\frac{1}{2}a+n+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left\lfloor\frac{i}{2}\right\rfloor\right)} \right\} \tag{3.5}$$

for $i = 0, 1, 2, 3, 4, 5$.

The coefficients A_i and B_i can be obtained from table 1 by changing b to $-n$.

For $i = 0$, (3.5) reduces to Dixon theorem (1.6).

For $i = 1, 2, 3$, (3.5) reduces to the results recorded in [6, equation (2), p 268] obtained by other means.

(b) In (2.1), if we take $y = -1$, $j = 0$, $G = 2$, $H = 1$, $f = 1 + a$, $g_1 = a$, $g_2 = d$ and $h_1 = 1 + a - d + i$, then for $i = 0, 1, 2, 3$, we will have

$$ {}_4F_3\left[\begin{matrix} a, & b, & c, & d \\ & 1+a-b, & 1+a-c, & 1+a-d+i \end{matrix} ; -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(1+a)_n n!} \times {}_3F_2\left[\begin{matrix} -n, & a, & d \\ & 1+a+n, & 1+a-d+i \end{matrix} ; 1\right].$$

Applying the generalized Dixon theorem (1.15), and simplifying, we get

$$ {}_4F_3\left[\begin{matrix} a, & b, & c, & d \\ & 1+a-b, & 1+a-c, & 1+a-d+i \end{matrix} ; -1\right] = \frac{2^{-2d+i}\Gamma(d-i)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d+i)}{\Gamma(1+a)\Gamma(d)\Gamma(1+a-b-c)\Gamma(1+a-2d+i)} \times \sum_{n=0}^{\infty} \frac{(b)_n (c)_n \Gamma(1+a+n)}{(1+a)_n n! \Gamma(1+a-d+n+i)} \times \left\{ A_i \frac{\Gamma\left(\frac{1}{2}a-d+\frac{1}{2}+\left\lfloor\frac{i+1}{2}\right\rfloor\right)\Gamma\left(\frac{1}{2}a+n-d+1+\left\lfloor\frac{i+1}{2}\right\rfloor\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+n+1\right)} + B_i \frac{\Gamma\left(\frac{1}{2}a-d+1+\left\lfloor\frac{i}{2}\right\rfloor\right)\Gamma\left(\frac{1}{2}a+n-d+\frac{3}{2}+\left\lfloor\frac{i}{2}\right\rfloor\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a+n+\frac{1}{2}\right)} \right\} \tag{3.6}$$

for $i = 0, 1, 2, 3$.

The coefficients A_i and B_i can be obtained from table 2 by changing b to $-n$ and c to d.

In (3.6), if we take $i = 0$, using the Dixon theorem (1.6) and simplifying, we obtain a known result recorded in [12, equation (1), p 560].

In (3.6), if we take $i = 1$ and values of A_i and B_i from table 2, use (1.15) (for $i = 1$), and simplify, we obtain the following new result:

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & d \\ & 1+a-b, & 1+a-c, & 2+a-d \end{matrix} ; -1\right]
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(d-1)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(d)}
\times \left\{ \left(\frac{1}{2}a\right) {}_3F_2\left[\begin{matrix} b, & c, & \frac{1}{2}a-d+\frac{3}{2} \\ & a-d+2, & \frac{1}{2}a+\frac{1}{2} \end{matrix} ; 1\right]
- \left(\frac{1}{2}a-d+1\right) {}_3F_2\left[\begin{matrix} b, & c, & \frac{1}{2}a-d+2 \\ & a-d+2, & \frac{1}{2}a+1 \end{matrix} ; 1\right]\right\}.
$$

Further to this, if we take $d = 1 + \frac{1}{2}a$, we get a known result [12, equation (2), p 546].

(c) In (2.1), if we take $y = -1$, $G = 1$, $H = 0$, $g_1 = a$ and $f = 1 + d$ then for $i = j = 0$, we have

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+d-b, & 1+d-c \end{matrix} ; -1\right]
= \frac{\Gamma(1+d-b)\Gamma(1+d-c)}{\Gamma(1+d)\Gamma(1+d-b-c)} \sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{(1+d)_n n!}
\times {}_2F_1\left[\begin{matrix} -n, & a \\ & 1+d+n \end{matrix} ; 1\right].
$$

Using the Gauss theorem, the duplication formula, and simplifying we obtain the following new transformation:

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+d-b, & 1+d-c \end{matrix} ; -1\right]
= \frac{\Gamma(1+d-b)\Gamma(1+d-c)}{\Gamma(1+d)\Gamma(1+d-b-c)}
\times {}_4F_3\left[\begin{matrix} b, & c, & \frac{1}{2}+\frac{1}{2}d-\frac{1}{2}a, & 1+\frac{1}{2}d-\frac{1}{2}a \\ & 1+d-a, & \frac{1}{2}+\frac{1}{2}d, & 1+\frac{1}{2}d \end{matrix} ; 1\right]. \tag{3.7}
$$

In this, if we take $d = a$, we obtain a known result [12, equation (2), p 546].

(d) In (2.1), if we take $G = 1$, $H = 0$, $g_1 = a$, $f = 1 + d$, $c = 1 + \frac{1}{2}d$ and $y = -1$, then for $i = j = 0$, we have

$$
{}_3F_2\left[\begin{matrix} a, & b, & 1+\frac{1}{2}d \\ & 1+d-b, & \frac{1}{2}d \end{matrix} ; -1\right] = \frac{\Gamma(1+d-b)\Gamma\left(\frac{1}{2}d\right)}{\Gamma(1+d)\Gamma\left(\frac{1}{2}-d\right)}
$$

$$\times \sum_{n=0}^{\infty} \frac{(b)_n\left(1+\frac{1}{2}d\right)_n}{(1+d)_n n!}\, {}_2F_1\left[\begin{matrix} -n, & a \\ & 1+d+n \end{matrix} ; 1\right].$$

Using the Gauss theorem, the duplication formula, and simplifying, we obtain the following new transformation:

$$ {}_3F_2\left[\begin{matrix} a, & b, & 1+\frac{1}{2}d \\ & 1+d-b, & \frac{1}{2}d \end{matrix} ; -1\right] = \frac{\Gamma\left(\frac{1}{2}d\right)\Gamma(1+d-b)}{\Gamma(1+d)\Gamma\left(\frac{1}{2}d-b\right)}$$
$$\times\, {}_3F_2\left[\begin{matrix} b, & \frac{1}{2}+\frac{1}{2}d-\frac{1}{2}a, & 1+\frac{1}{2}d-\frac{1}{2}a \\ & 1+d-a, & \frac{1}{2}+\frac{1}{2}d \end{matrix} ; 1\right]. \qquad (3.8)$$

In this, if we take $d = a$ and use the Gauss theorem, we get the known result [12, equation (7), p 547].

Remark. The result (3.8) can also be obtained from (3.7) by taking $c = 1 + \frac{1}{2}d$.

(e) In (2.1), if we take $G = 2$, $H = 1$, $y = 1$, $g_1 = a$, $g_2 = 1 + \frac{1}{2}a$, $h_1 = \frac{1}{2}a$ and $f = 1 + a$, then for $i = j = 0$, we have

$$ {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix} ; 1\right]$$
$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{(1+a)_n n!}$$
$$\times\, {}_3F_2\left[\begin{matrix} a, & 1+\frac{1}{2}a, & -n \\ & 1+a+n, & \frac{1}{2}a \end{matrix} ; -1\right]. \qquad (3.9)$$

Using [12, equation (17), p 547], we will have

$$ {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix} ; 1\right]$$
$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{\left(\frac{1}{2}+\frac{1}{2}a\right)_n n!}.$$

Summing the series and using the Gauss summation theorem, we finally obtain a known result [14, equation (III.22), p 245].

(f) In (2.1), if we take $G = 3$, $H = 2$, $y = 1$, $g_1 = a$, $g_2 = 1 + \frac{1}{2}a$, $h_1 = 1 + \frac{1}{2}a$, $h_2 = \frac{1}{2}a$, $f = 1 + a$, $g_3 = d$ and $h_3 = 1 + a - d$, then for $i = j = 0$, we have

$$ {}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix} ; 1\right]$$
$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{(1+a)_n n!}$$
$$\times\, {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & d, & -n \\ & \frac{1}{2}a, & 1+a+n, & 1+a-d \end{matrix} ; -1\right].$$

Using [14, equation (III.11), p 244], we get

$$
{}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix} ; 1\right]
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{(1+a-d)_n n!}. \tag{3.10}
$$

Summing the series and using the Gauss summation theorem, we finally obtain a known result [14, equation (III.12), p 244].

(g) In (2.1), if we take $G = 3$, $H = 2$, $y = -1$, $g_1 = a$, $g_2 = 1+\frac{1}{2}a$, $g_3 = d$, $h_1 = \frac{1}{2}a$, $h_2 = 1+a-d$ and $f = 1+a$, then for $i = j = 0$, we have

$$
{}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix} ; -1\right]
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}
\times {}_3F_2\left[\begin{matrix} b, & c, & \frac{1}{2}+\frac{1}{2}a-d \\ & \frac{1}{2}+\frac{1}{2}a, & 1+a-d \end{matrix} ; 1\right]. \tag{3.11}
$$

In (3.11), if we take $b = -m$, then

$$
{}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & -m \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a+m \end{matrix} ; -1\right]
= \frac{(1+a)_m}{(1+a-c)_m}\, {}_3F_2\left[\begin{matrix} -m, & c, & \frac{1}{2}+\frac{1}{2}a-d \\ & \frac{1}{2}+\frac{1}{2}a, & 1+a-d \end{matrix} ; 1\right]. \tag{3.12}
$$

(h) In (2.1), if we take $G = 4$, $H = 3$, $y = -1$, $g_1 = a$, $g_2 = d$, $g_3 = e$, $g_4 = 1+\frac{1}{2}a$, $h_1 = 1+a-d$, $h_2 = 1+a-e$, $h_3 = \frac{1}{2}a$ and $f = 1+a$, then for $i = j = 0$, we have

$$
{}_6F_5\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix} ; -1\right]
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{(1+a)_n n!}
\times {}_5F_4\left[\begin{matrix} -n, & a, & d, & e, & 1+\frac{1}{2}a \\ & 1+a+n, & 1+a-d, & 1+a-e, & \frac{1}{2}a \end{matrix} ; 1\right].
$$

Using [1, equation (3), p 25] and summing the series, we get a known result [1, equation (2), p 28].

(i) In (2.1), if we take $G = 4$, $H = 3$, $y = 1$, $g_1 = a$, $g_2 = d$, $g_3 = e$, $g_4 = 1+\frac{1}{2}a$, $h_1 = 1+a-d$, $h_2 = 1+a-e$, $h_3 = 1+\frac{1}{2}a$ and $f = 1+a$, then for $i = j = 0$, we have

$$
{}_6F_5\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e & \end{matrix} ; 1\right]
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{(1+a)_n n!}
$$

$$
\times {}_5F_4\left[\begin{matrix} -n, & a, & 1+\frac{1}{2}a, & d, & e \\ \frac{1}{2}a, & 1+a+n, & 1+a-d, & 1+a-e & \end{matrix} ; -1\right].
$$

Using (3.12), we obtain

$$
{}_6F_5\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e & \end{matrix} ; 1\right]
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{(1+a-d)_n n!}
$$

$$
\times {}_3F_2\left[\begin{matrix} -n, & d, & \frac{1}{2}+\frac{1}{2}a-e \\ \frac{1}{2}+\frac{1}{2}a, & 1+a-e & \end{matrix} ; 1\right].
$$

This again appears to be a new transformation.

4. Application of the new transformation (3.3)

In this section, we shall establish the following new transformation:

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & d+1 \\ 1+a-b, & 1+a-c, & d & \end{matrix} ; -1\right]
$$

$$
= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} + \left(2-\frac{a}{d}\right) \frac{bc\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(a+3)\Gamma(1+a-b-c)}
$$

$$
\times {}_4F_3\left[\begin{matrix} 1, & \frac{3}{2}, & b+1, & c+1 \\ 2, & \frac{1}{2}a+\frac{3}{2}, & \frac{1}{2}a+2 & \end{matrix} ; 1\right]. \tag{4.1}
$$

Proof. In order to derive (4.1), we shall use the following result, which can be derived easily, and hence given here without proof

$$
{}_3F_2\left[\begin{matrix} b, & c, & \frac{1}{2} \\ \frac{1}{2}a+\frac{1}{2}, & \frac{1}{2}a+1 & \end{matrix} ; 1\right] = 1 + \frac{2bc}{(a+1)(a+2)}
$$

$$
\times {}_4F_3\left[\begin{matrix} b+1, & c+1, & \frac{3}{2}, & 1 \\ 2, & \frac{1}{2}a+\frac{3}{2}, & \frac{1}{2}a+2 & \end{matrix} ; 1\right]. \tag{4.2}
$$

Starting with the left-hand side of (4.1), denoting it by S, expressing ${}_4F_3$ as a series, we have

$$
S = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n(-1)^n}{(1+a-b)_n(1+a-c)_n n!} \left\{\frac{(d+1)_n}{(d)_n}\right\}.
$$

Noting that $\frac{(d+1)_n}{(d)_n} = 1 + \frac{n}{d}$, we have

$$S = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n (-1)^n}{(1+a-b)_n (1+a-c)_n n!} \left\{1 + \frac{n}{d}\right\}.$$

Separating the above series into two parts, adjusting in the second series, and summing, we obtain

$$S = {}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+a-b, & 1+a-c \end{matrix} ; -1\right] - \frac{abc}{d(1+a-b)(1+a-c)}$$
$$\times {}_3F_2\left[\begin{matrix} a+1, & b+1, & c+1 \\ & 2+a-b, & 2+a-c \end{matrix} ; -1\right].$$

The first and second ${}_3F_2$ can be evaluated with the help of the known result [12, equation (2), p 546] and (3.3), respectively. This gives

$$S = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}\, {}_3F_2\left[\begin{matrix} b, & c, & \frac{1}{2} \\ \frac{1}{2}+\frac{1}{2}a, & 1+\frac{1}{2}a & \end{matrix} ; 1\right]$$
$$- \frac{abc}{d(1+a-b)(1+a-c)} \frac{\Gamma(2+a-b)\Gamma(2+a-c)}{\Gamma(3+a)\Gamma(1+a-b-c)}$$
$$\times {}_4F_3\left[\begin{matrix} b+1, & c+1, & 1, & \frac{3}{2} \\ & 2, & \frac{3}{2}+\frac{1}{2}a, & 2+\frac{1}{2}a \end{matrix} ; 1\right].$$

Using (4.2), we finally have

$$S = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} + \left(2 - \frac{a}{d}\right) \frac{bc\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(a+3)\Gamma(1+a-b-c)}$$
$$\times {}_4F_3\left[\begin{matrix} 1, & \frac{3}{2}, & b+1, & c+1 \\ & 2, & \frac{1}{2}a+\frac{3}{2}, & \frac{1}{2}a+2 \end{matrix} ; 1\right].$$

This completes the proof of (4.1). □

4.1. Special case

In (4.1), if we take $d = \frac{1}{2}a$, we obtain

$${}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix} ; -1\right]$$
$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}$$

which is a known result [12, equation (2), p 560].

5. Concluding remark

According to the recent review of Milgram [11], 'The results of Lavoie *et al* [5–7] are limited to small subset of near-diagonal contiguous cases whereas Lewanowicz has independently given a general result valid for all off-diagonal distances of the generalized Watson theorem'. By application of Thomae identities this remark extends to the case of the generalized Dixon and Whipple theorems. Consequently, our equations (1.14), (1.15), (3.6), etc, can easily be extended to any integer i.

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